Extending curves in metric spaces II: bi-Lipschitz arcs

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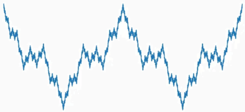

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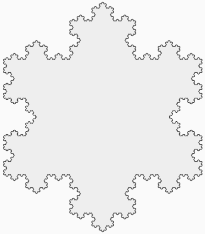

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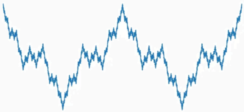
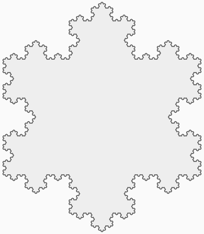
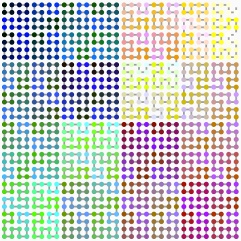

Sources: https://medium.com/@Melearning/weierstrass-function-in-python-6b1e6819df3a, Wikimedia commons

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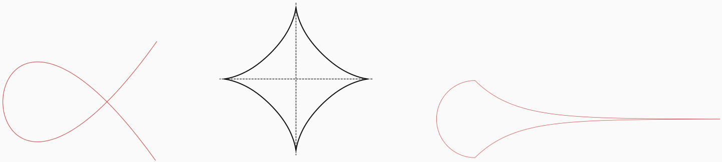

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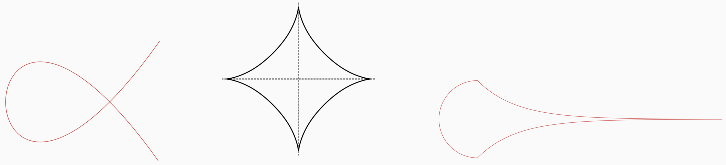

A set $\Gamma \subset X$ is a **bi-Lipschitz curve** if γ can be chosen to be bi-Lipschitz.

$$\frac{1}{L}|a-b| \leq d(\gamma(a),\gamma(b)) \leq L|a-b| \text{ for all } a, b \in I$$

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If $f : X \to Y$ is bi-Lipschitz, then

- X is complete if and only if Y is.
- X is bounded if and only if Y is.
- X and Y have the same Hausdorff dimension.
- X is Ahlfors Q-regular if and only if Y is.
- X supports a p-Poincaré inequality if and only if Y does.
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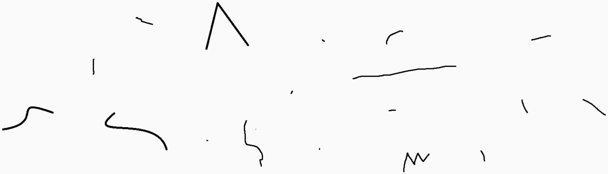
From the perspective of metric geometry,
a bi-Lipschitz curve Γ is "pretty much the same" as an interval in $\mathbb{R}$.

Our question

Fix a metric space Y, a set $A \subset \mathbb{R}$, and an L-bi-Lipschitz map $f : A \to Y$.



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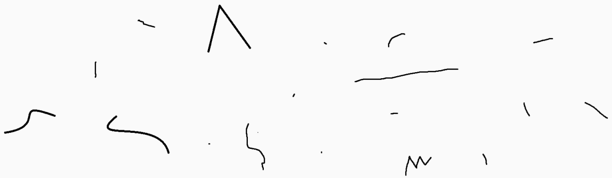

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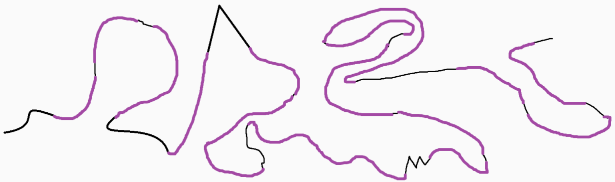

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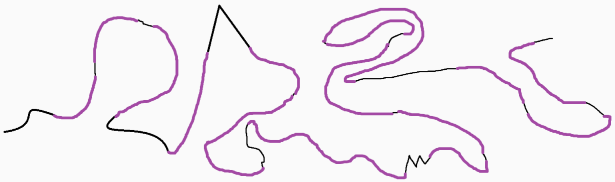

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We'll say that Y has the bi-Lipschitz extension property for curves if, for all $L > 0$, there is a constant $C(L, Y) \geq 1$ satisfying the following: for any $A \subset \mathbb{R}$ and any L-bi-Lipschitz map $f : A \to Y$, there is a C-bi-Lipschitz curve $F : [\inf A, \sup A] \to Y$ with $F|_A = f$.

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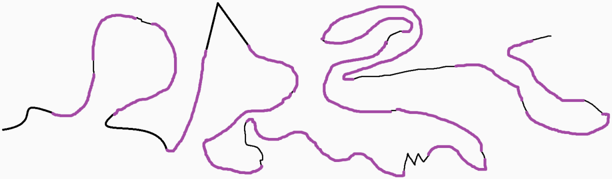

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This is not obvious even when $Y = \mathbb{R}^n$!

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Theorem (MacManus, 1995)

$\mathbb{R}^2$ *has the bi-Lipschitz extension property for curves.*

Bi-Lipschitz curve extensions in metric spaces

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Some metric spaces satisfying these assumptions:

- Riemannian manifolds with Ricci curvature bounded below,
- orientable, n-regular, linearly locally contractible n-manifolds with $n \geq 3$,
- Carnot groups (which include Euclidean spaces and the Heisenberg group),
- certain hyperbolic buildings,
- Laakso spaces,
- certain Menger sponges.

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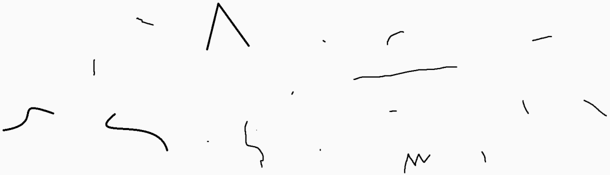

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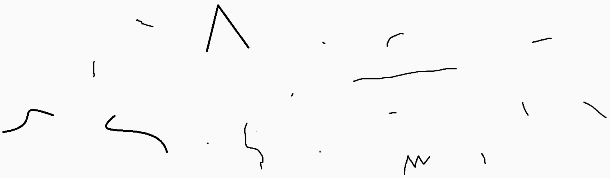

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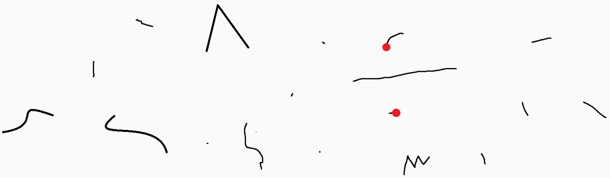

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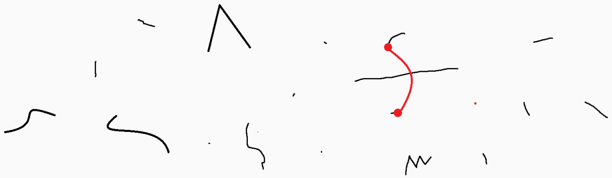

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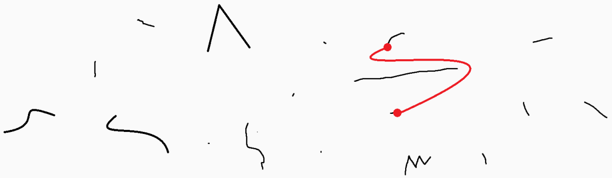

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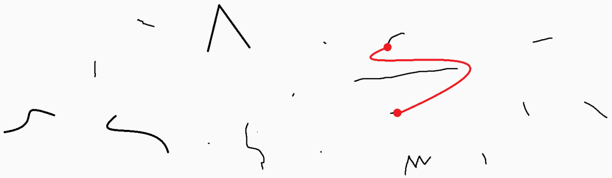

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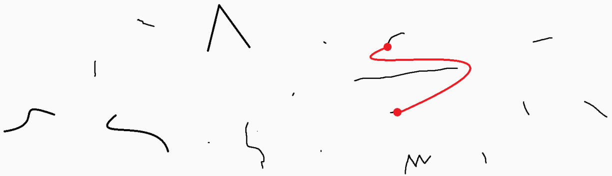

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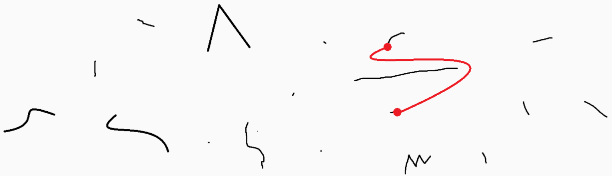

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2. We need a lot of curves to choose from.

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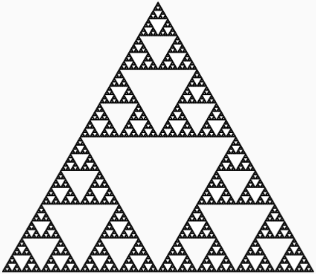

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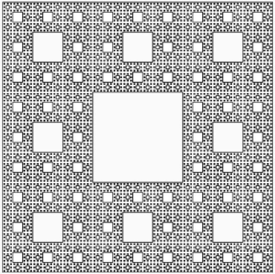

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Facts:

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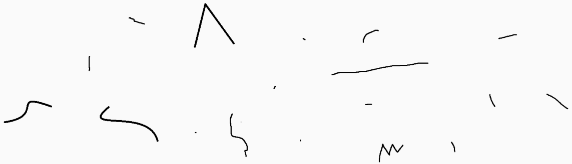

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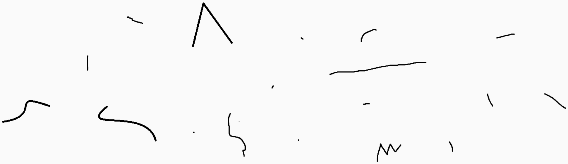

$f(A)$ is at most 1-dimensional within a space of dimension $Q > 2$!

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Actual definition:

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$$\frac{1}{\mu(B(x,r))} \int_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \lesssim r \left(\frac{1}{\mu(B(x,r))} \int_{B(x,r)} g^p \, d\mu \right)^{1/p}$$

where $g : Y \to [0, \infty)$ is an upper gradient of u:
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(For the experts: $\lambda = 1$ since we can assume Y is geodesic and doubling.)

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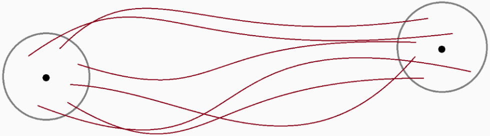

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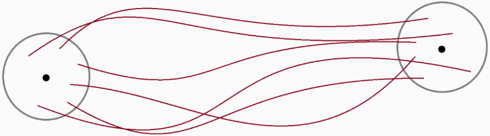

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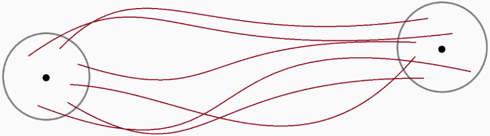

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$$\mathrm{Mod}_p(\Gamma) = \inf \left\{ \int_Y \rho^p \, d\mu \ : \ \rho : Y \to [0, \infty) \text{ with } \int_\gamma \rho \, ds \geq 1 \text{ for } \gamma \in \Gamma \right\}$$

Modulus of curves

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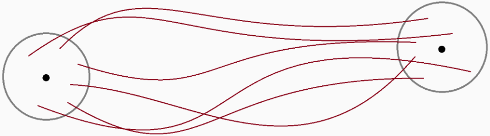
What does "a lot of curves" mean? Suppose Γ is a family of curves in Y.



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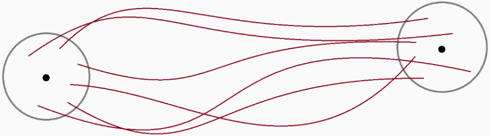

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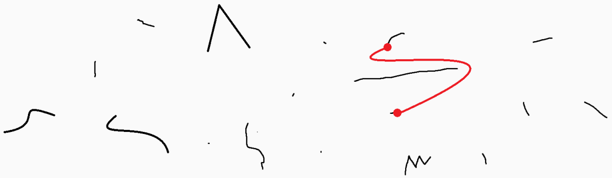

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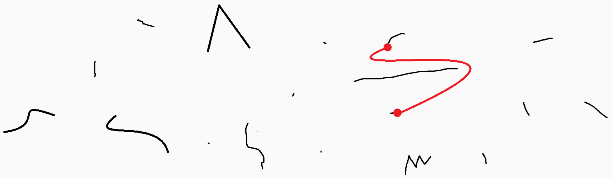

More precisely, we want the p-modulus of the family of curves connecting any two points and missing $f(A)$ to be bounded away from 0.

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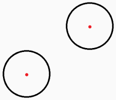
How does the p-Poincaré inequality help with this?

Combining ideas

Pick two points $x, y \in Y$ and two balls B_x and B_y of radius $r = \frac{d(x,y)}{3}$.



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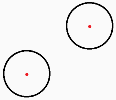

Set Γ to be the collection of all curves starting in B_x, ending in B_y, and staying within a ball B of radius $6r$.

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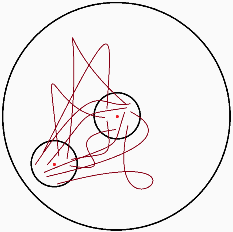

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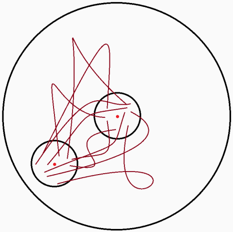

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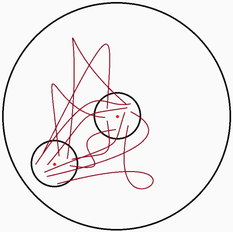

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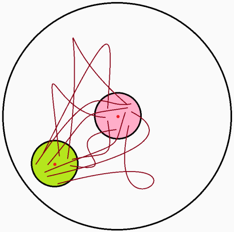

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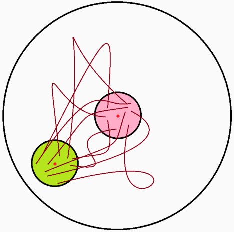

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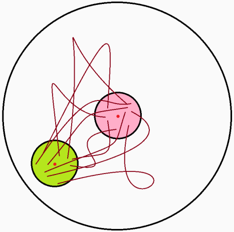

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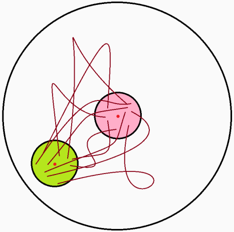

p-Poincaré inequality:

$$\frac{1}{\mu(B)^p}\left(\int_B |u - u_B|\, d\mu\right)^p \lesssim \frac{r^p}{\mu(B)} \int_B g_u^p\, d\mu$$

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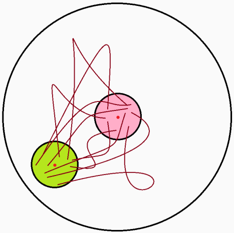

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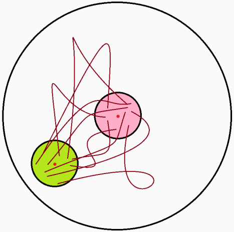

p-**Poincaré inequality:**

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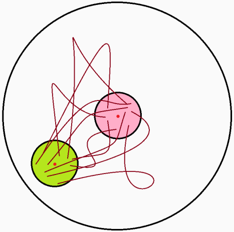

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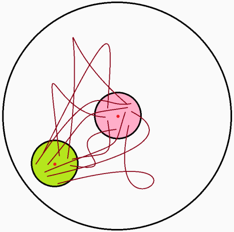

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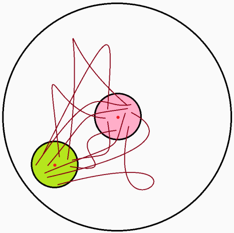

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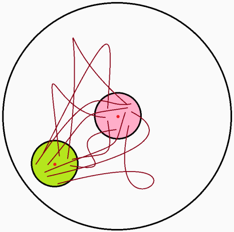

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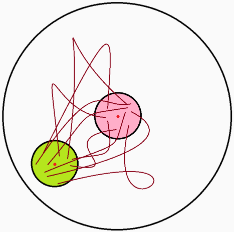

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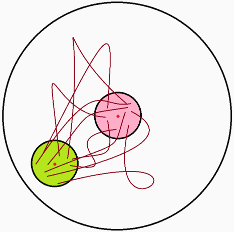

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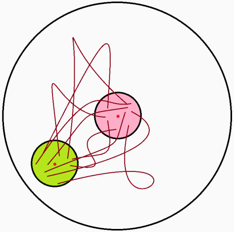

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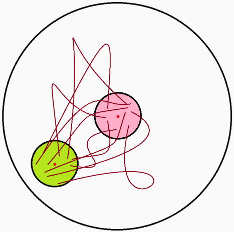

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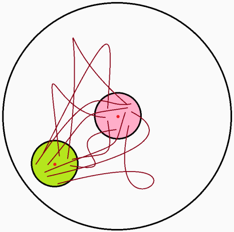

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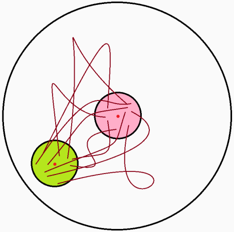

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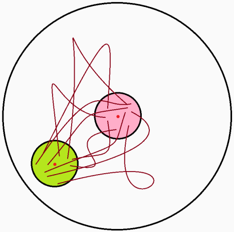

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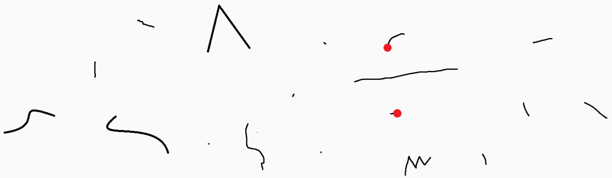

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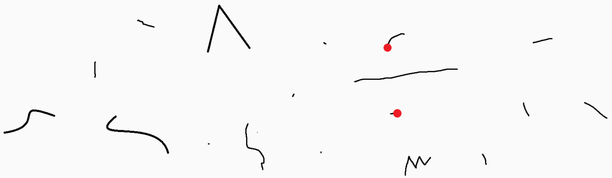

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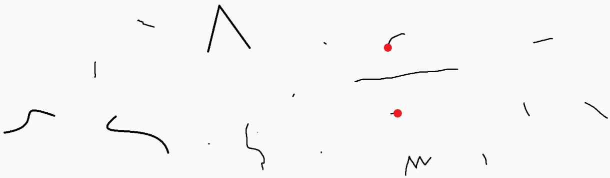

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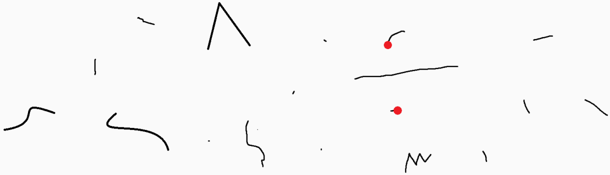

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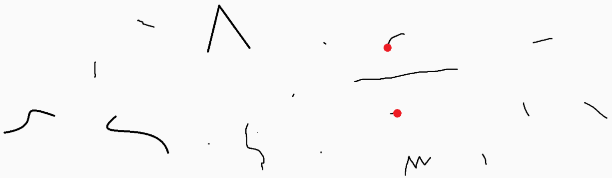

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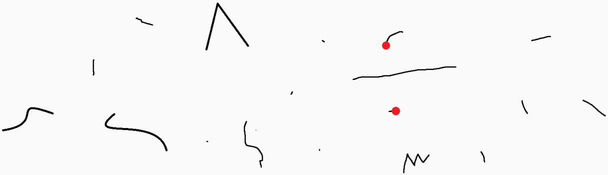
So we can always find the curve we're looking for!

Main tool



Theorem (Honeycutt-Vellis-Z, 2024)

Suppose (Y, d, μ) is a Q-Ahlfors regular, geodesic metric measure space supporting a p-Poincaré inequality for $1 < p < Q$ and suppose $f : \mathbb{R} \supset A \to Y$ is bi-Lipschitz.

Then, for any $x, y \in Y \setminus f(A)$ with $d(x, y) \approx dist(\{x, y\}, f(A))$, there is a curve $\gamma : [0, 1] \to Y$ from x to y such that

- *$length(\gamma) \lesssim d(x, y)$*
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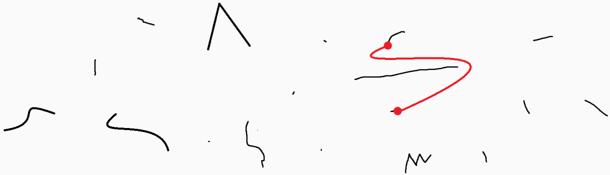

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The second ingredient

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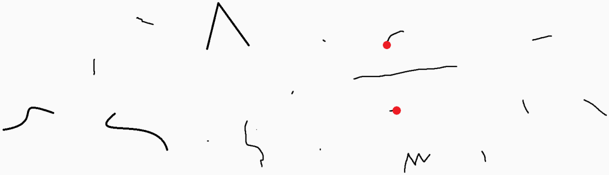

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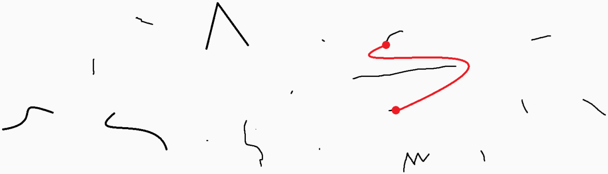

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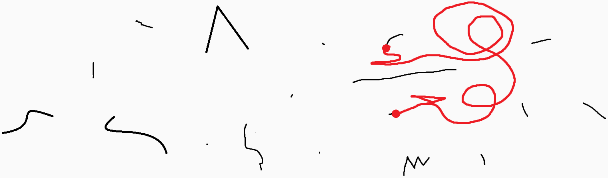

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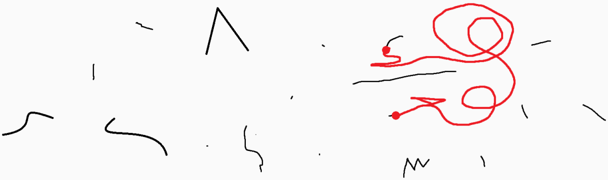

Proposition (Lytchak-Wenger 2020; Honeycutt-Vellis-Z 2024)

Suppose (Y, d) is a doubling, geodesic metric space.

If $\gamma : [0,1] \to Y$ is a curve from x to y of diameter 1, then, for any $\varepsilon > 0$ there is an $L(\varepsilon)$-bi-Lipschitz curve $\eta : [0,1] \to Y$ from x to y with

$$\mathit{dist}(\eta(t), \gamma([0,1])) \leq \varepsilon.$$

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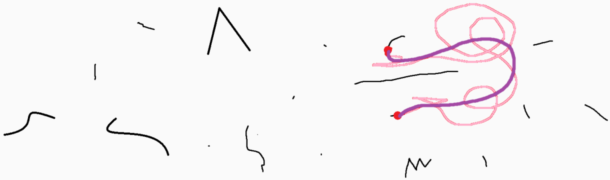

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If $f(a)$ *is a closest point in* $f([0, 1])$ *to* y*, then the concatenation of* $f|_{[0,a]}$ *with a geodesic from* $f(a)$ *to* y *is* $2L$*-bi-Lipschitz.*

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Lemma

Suppose (Y, d) *is a geodesic metric space,* $f : [0,1] \to Y$ *is* L*-bi-Lipschitz, and* $y \in Y$*.*
If $f(a)$ *is a closest point in* $f([0,1])$ *to* y*, then the concatenation of* $f|_{[0,a]}$ *with a geodesic from* $f(a)$ *to* y *is* $2L$*-bi-Lipschitz.*

Sharpness of assumptions

Assumption 2.

Assumption 2. Y supports p-Poincaré inequality for some $1 < p < Q - 1$.

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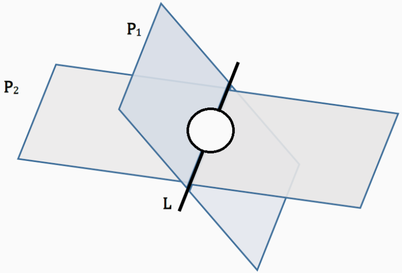
Suppose $Q \geq 2$ is an integer and P_1 and P_2 are Q-planes intersecting in a line. Fix x on the line, and set $Y = P_1 \cup P_2 \setminus B(x, 1)$.



Then Y is complete, Q-Ahlfors regular, and supports a p-Poincaré inequality for all $p > Q - 1$.

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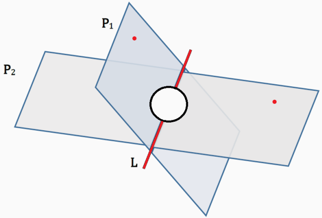
Define $f : (-\infty, -1] \cup \{-\frac{1}{2}, \frac{1}{2}\} \cup [1, \infty) \to Y$ as follows:
$f(-\frac{1}{2})$ and $f(\frac{1}{2})$ lie in P_1 and P_2 respectively but away from the line, and f sends $(-\infty, -1] \cup [1, \infty)$ to the line isometrically.



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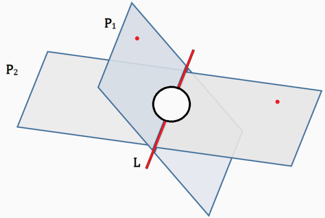

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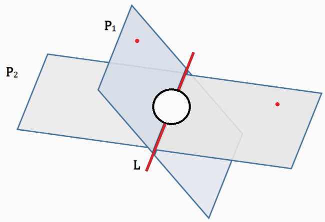

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Since Y satisfies A 1., this shows that A 2. can't be removed.

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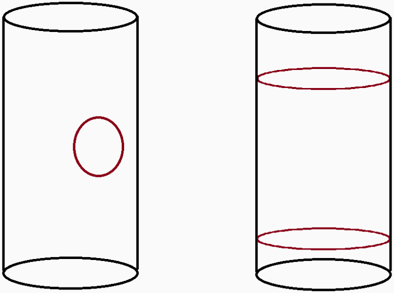
The infinite cylinder $Y = \mathbb{S}^2 \times \mathbb{R}$ with the length metric is not Ahlfors regular for any Q.



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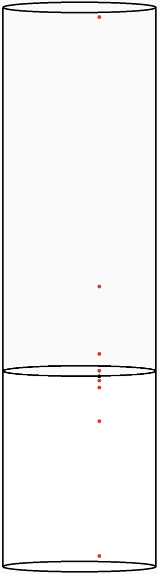

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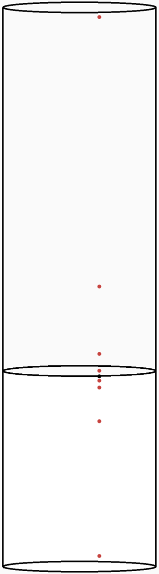

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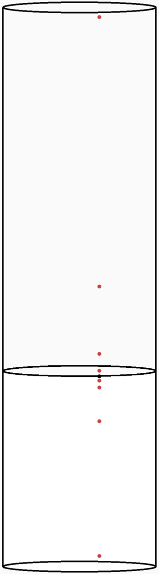

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Theorem (MacManus, 1995)

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Open question: If Y is 2-Ahlfors regular and supports a 1-Poincaré inequality, what else must be true for it to have the bi-Lipschitz extension property for curves?

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Corollary (HVZ, 2024)

Let Y be a complete Q-Ahlfors regular metric measure space supporting a p-Poincaré inequality for some $1 < p \leq Q - 1$. If $E \subset Y$ has Assouad dimension less than 1, then E is bi-Lipschitz rectifiable.

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