

An introduction to Carnot Groups

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June 2026

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THE OHIO STATE
UNIVERSITY

Riding on a unicycle

An example

Imagine a person standing in a room facing a given direction.



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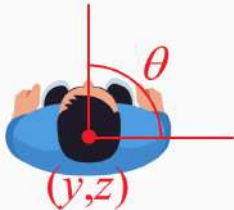


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- 2 variables for their location: y and z
- 1 variable for their direction: θ

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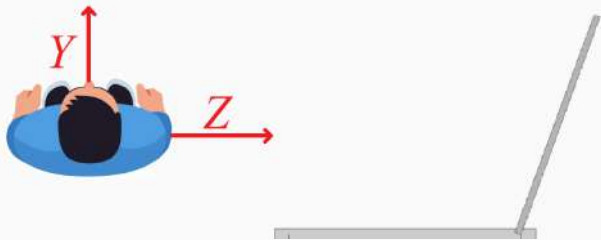
Let's represent any such configuration as a triple (θ, y, z) .

An example



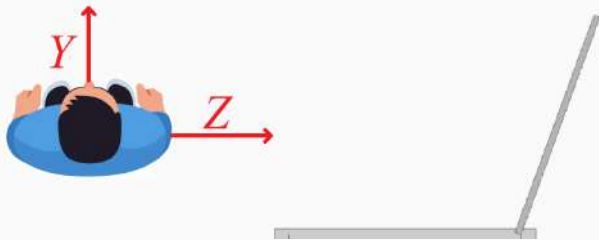
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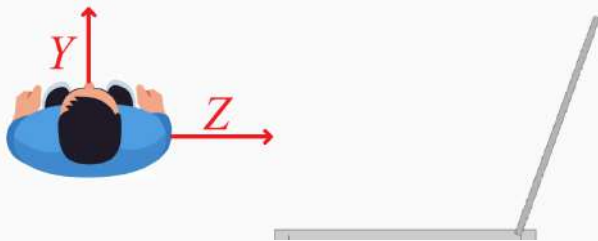
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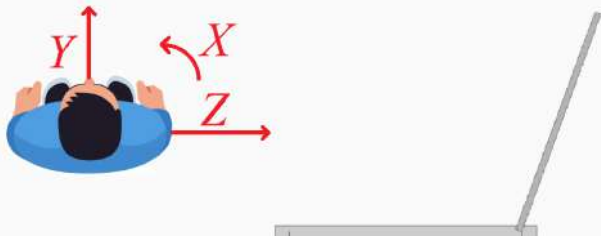
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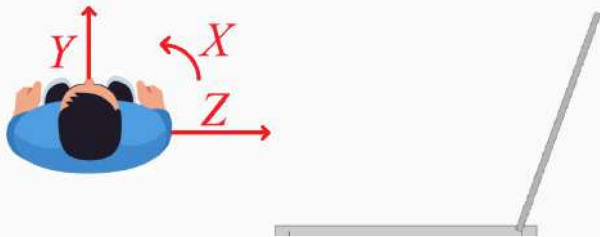
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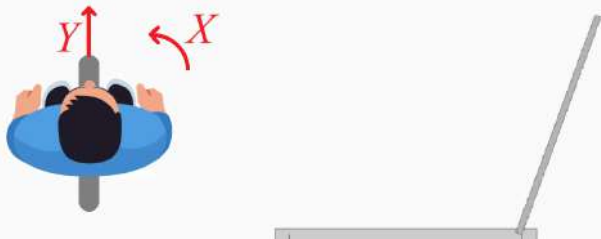
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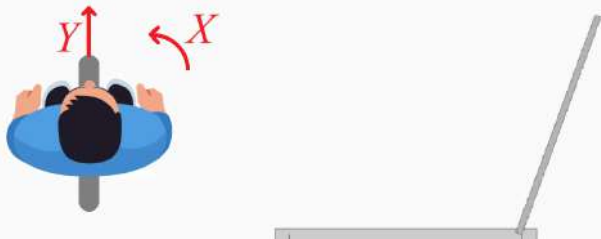
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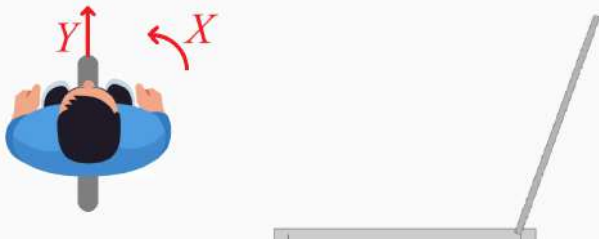
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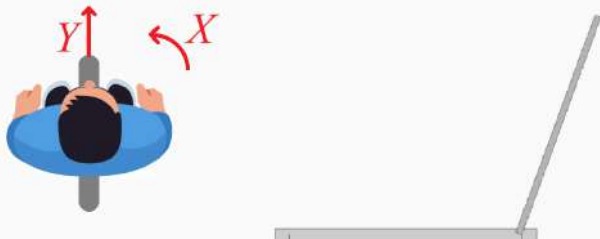


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Wherever they are, they can only move in 2 “directions.”
But they can still get to any 3 point configuration (θ, y, z) !

Carnot groups for analysts

Define a *Lie group* operation $*$ on \mathbb{R}^N , and write

$$\mathbb{R}^N = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \dots \times \mathbb{R}^{n_s}.$$

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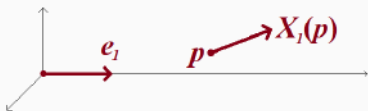
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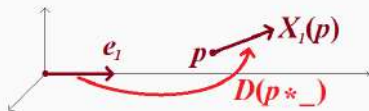
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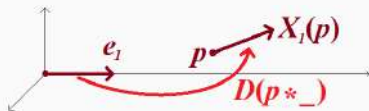
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Then any left invariant vector field on \mathbb{R}^N can be written as a linear combination of Lie brackets of X_1, \dots, X_{n_1} .

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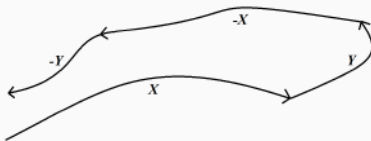
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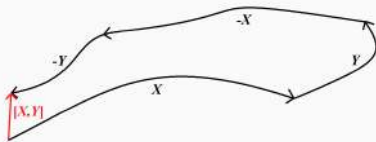
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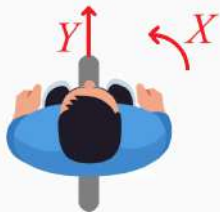
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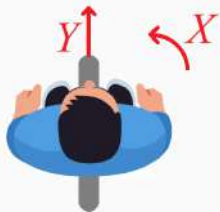
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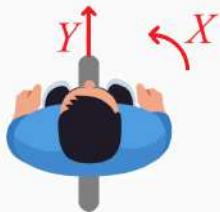
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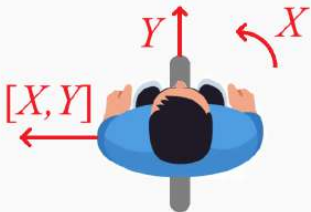
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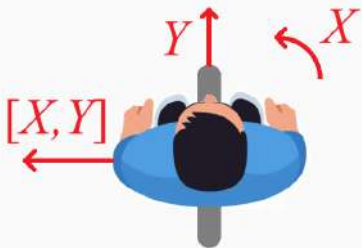
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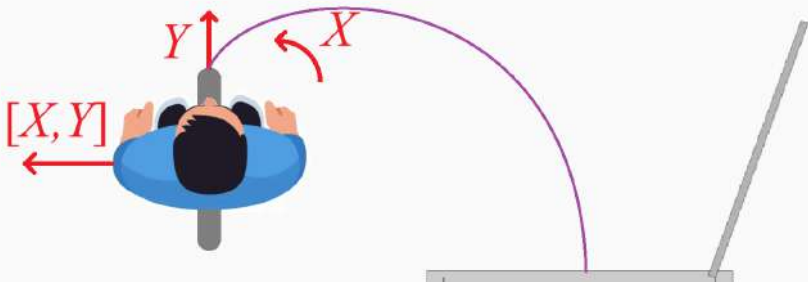
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Fact: **H** is a Carnot group.

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Fact: **H** is a Carnot group. In particular,

$$[X, Y] = XY - YX = \frac{\partial}{\partial z}.$$

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Measurables in quantum physics are represented by operators.

E.g. the **position operator** \hat{x} is defined as $\hat{x}(\psi) = x\psi$
(since $\mathbb{E}[\hat{x}\psi] = \mathbb{E}[x|\psi|^2]$).

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“Your talk had too much bull****.” – Piotr Hajłasz

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We are only allowed to move in a direction that is a *linear combination* of $X(p)$ and $Y(p)$.

Horizontal curves

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We are only allowed to move in a direction that is a *linear combination* of $X(p)$ and $Y(p)$.

At each point p , we can only move in a **plane** of directions!

(Call this plane H_p the **horizontal plane** at p .)

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Length

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Fix two points p and q in the Heisenberg group.

How long does it take to travel from p to q along a horizontal path?

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We defined the speed $|\gamma'(t)|$ to be $\sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}$ because

$$\gamma'(t) = f'(t)\hat{\mathbf{i}} + g'(t)\hat{\mathbf{j}} + h'(t)\hat{\mathbf{k}},$$

and the vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are the building blocks of \mathbb{R}^3 .

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Geodesics

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If so, what do they look like?

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This is smallest when $f'(t) \equiv 0$ and $g'(t) \equiv 0$.

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These facts are true for **any** horizontal curve from $(0, 0, 0)$ to $(0, 0, \alpha)$!

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Understanding the Heisenberg group

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Theorem

A horizontal curve from $(0, 0, 0)$ to a point on the z -axis is a geodesic if and only if its projection to the xy -plane is a circle.

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Suppose $\gamma : [0, 1] \rightarrow \mathbf{H}$ is Lipschitz. Then γ is horizontal.

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Fix $t \in [0, 1]$ and $h > 0$.

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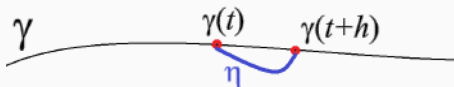
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We need to show

$$\frac{\gamma_3(t+h) - \gamma_3(t)}{h} - \gamma_1(t) \frac{\gamma_2(t+h) - \gamma_2(t)}{h} = o(h).$$

Proof that $\gamma'_3 = \gamma_1 \gamma'_2$ a.e.

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$$\begin{aligned} & \left| \frac{\gamma_3(t+h) - \gamma_3(t)}{h} - \gamma_1(t) \frac{\gamma_2(t+h) - \gamma_2(t)}{h} \right| \\ &= \left| \frac{\eta_3(1) - \eta_3(0)}{h} - \eta_1(0) \frac{\eta_2(1) - \eta_2(0)}{h} \right| \\ &= \left| \frac{1}{h} \int_0^1 \eta'_3 - \frac{1}{h} \int_0^1 \eta_1(0) \eta'_2 \right| \\ &= \left| \frac{1}{h} \int_0^1 \eta_1 \eta'_2 - \frac{1}{h} \int_0^1 \eta_1(0) \eta'_2 \right| \\ &\leq \frac{1}{h} \int_0^1 |\eta_1 - \eta_1(0)| |\eta'_2| \leq \frac{1}{h} \int_0^1 (\ell s) \cdot \ell \, ds \end{aligned}$$

Proof that $\gamma'_3 = \gamma_1 \gamma'_2$ a.e.

$$\begin{aligned} & \left| \frac{\gamma_3(t+h) - \gamma_3(t)}{h} - \gamma_1(t) \frac{\gamma_2(t+h) - \gamma_2(t)}{h} \right| \\ &= \left| \frac{\eta_3(1) - \eta_3(0)}{h} - \eta_1(0) \frac{\eta_2(1) - \eta_2(0)}{h} \right| \\ &= \left| \frac{1}{h} \int_0^1 \eta'_3 - \frac{1}{h} \int_0^1 \eta_1(0) \eta'_2 \right| \\ &= \left| \frac{1}{h} \int_0^1 \eta_1 \eta'_2 - \frac{1}{h} \int_0^1 \eta_1(0) \eta'_2 \right| \\ &\leq \frac{1}{h} \int_0^1 |\eta_1 - \eta_1(0)| |\eta'_2| \leq \frac{1}{h} \int_0^1 (\ell s) \cdot \ell \, ds \lesssim \frac{\ell}{h} \cdot \ell \end{aligned}$$

Proof that $\gamma'_3 = \gamma_1 \gamma'_2$ a.e.

$$\begin{aligned} & \left| \frac{\gamma_3(t+h) - \gamma_3(t)}{h} - \gamma_1(t) \frac{\gamma_2(t+h) - \gamma_2(t)}{h} \right| \\ &= \left| \frac{\eta_3(1) - \eta_3(0)}{h} - \eta_1(0) \frac{\eta_2(1) - \eta_2(0)}{h} \right| \\ &= \left| \frac{1}{h} \int_0^1 \eta'_3 - \frac{1}{h} \int_0^1 \eta_1(0) \eta'_2 \right| \\ &= \left| \frac{1}{h} \int_0^1 \eta_1 \eta'_2 - \frac{1}{h} \int_0^1 \eta_1(0) \eta'_2 \right| \\ &\leq \frac{1}{h} \int_0^1 |\eta_1 - \eta_1(0)| |\eta'_2| \leq \frac{1}{h} \int_0^1 (\ell s) \cdot \ell \, ds \lesssim \frac{\ell}{h} \cdot \ell \\ &= \frac{d(\gamma(t+h), \gamma(t))}{h} \cdot d(\gamma(t+h), \gamma(t)) \end{aligned}$$

Proof that $\gamma'_3 = \gamma_1 \gamma'_2$ a.e.

$$\begin{aligned} & \left| \frac{\gamma_3(t+h) - \gamma_3(t)}{h} - \gamma_1(t) \frac{\gamma_2(t+h) - \gamma_2(t)}{h} \right| \\ &= \left| \frac{\eta_3(1) - \eta_3(0)}{h} - \eta_1(0) \frac{\eta_2(1) - \eta_2(0)}{h} \right| \\ &= \left| \frac{1}{h} \int_0^1 \eta'_3 - \frac{1}{h} \int_0^1 \eta_1(0) \eta'_2 \right| \\ &= \left| \frac{1}{h} \int_0^1 \eta_1 \eta'_2 - \frac{1}{h} \int_0^1 \eta_1(0) \eta'_2 \right| \\ &\leq \frac{1}{h} \int_0^1 |\eta_1 - \eta_1(0)| |\eta'_2| \leq \frac{1}{h} \int_0^1 (\ell s) \cdot \ell \, ds \lesssim \frac{\ell}{h} \cdot \ell \\ &= \frac{d(\gamma(t+h), \gamma(t))}{h} \cdot d(\gamma(t+h), \gamma(t)) \xrightarrow{\text{Kirchheim}} 0 \text{ a.e.} \end{aligned}$$

Proof that $\gamma'_3 = \gamma_1 \gamma'_2$ a.e.

$$\begin{aligned} & \left| \frac{\gamma_3(t+h) - \gamma_3(t)}{h} - \gamma_1(t) \frac{\gamma_2(t+h) - \gamma_2(t)}{h} \right| \\ &= \left| \frac{\eta_3(1) - \eta_3(0)}{h} - \eta_1(0) \frac{\eta_2(1) - \eta_2(0)}{h} \right| \\ &= \left| \frac{1}{h} \int_0^1 \eta'_3 - \frac{1}{h} \int_0^1 \eta_1(0) \eta'_2 \right| \\ &= \left| \frac{1}{h} \int_0^1 \eta_1 \eta'_2 - \frac{1}{h} \int_0^1 \eta_1(0) \eta'_2 \right| \\ &\leq \frac{1}{h} \int_0^1 |\eta_1 - \eta_1(0)| |\eta'_2| \leq \frac{1}{h} \int_0^1 (ls) \cdot \ell \, ds \lesssim \frac{\ell}{h} \cdot \ell \\ &= \frac{d(\gamma(t+h), \gamma(t))}{h} \cdot d(\gamma(t+h), \gamma(t)) \xrightarrow{\text{Kirchheim}} 0 \text{ a.e.} \end{aligned}$$

That is, γ'_3 exists and equals $\gamma_1 \gamma'_2$ a.e.

Proof that $\gamma'_3 = \gamma_1 \gamma'_2$ a.e.

$$\begin{aligned} & \left| \frac{\gamma_3(t+h) - \gamma_3(t)}{h} - \gamma_1(t) \frac{\gamma_2(t+h) - \gamma_2(t)}{h} \right| \\ &= \left| \frac{\eta_3(1) - \eta_3(0)}{h} - \eta_1(0) \frac{\eta_2(1) - \eta_2(0)}{h} \right| \\ &= \left| \frac{1}{h} \int_0^1 \eta'_3 - \frac{1}{h} \int_0^1 \eta_1(0) \eta'_2 \right| \\ &= \left| \frac{1}{h} \int_0^1 \eta_1 \eta'_2 - \frac{1}{h} \int_0^1 \eta_1(0) \eta'_2 \right| \\ &\leq \frac{1}{h} \int_0^1 |\eta_1 - \eta_1(0)| |\eta'_2| \leq \frac{1}{h} \int_0^1 (ls) \cdot \ell \, ds \lesssim \frac{\ell}{h} \cdot \ell \\ &= \frac{d(\gamma(t+h), \gamma(t))}{h} \cdot d(\gamma(t+h), \gamma(t)) \xrightarrow{\text{Kirchheim}} 0 \text{ a.e.} \end{aligned}$$

That is, γ'_3 exists and equals $\gamma_1 \gamma'_2$ a.e. □

Thank you!!

