Extending curves in metric spaces I: smooth curves in Carnot groups

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The Whitney Extension Theorem

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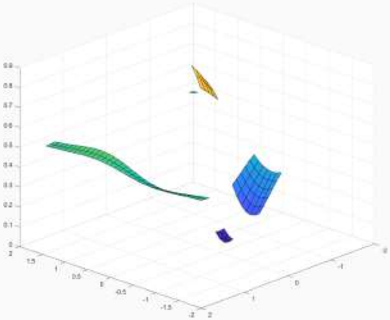

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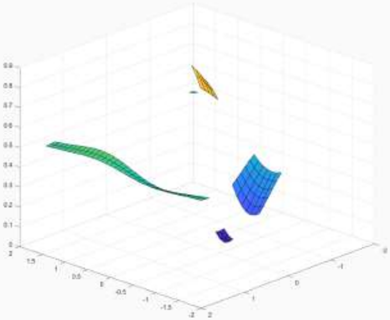

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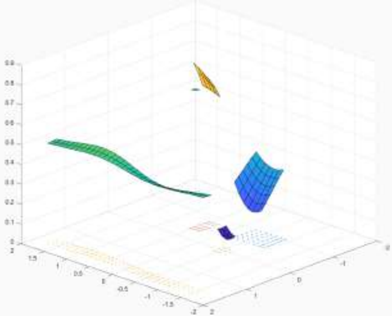

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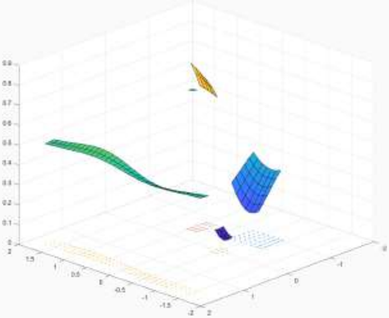

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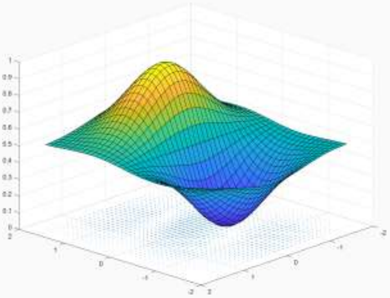

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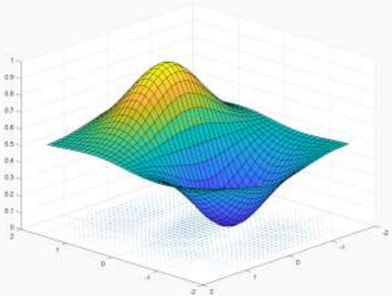

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Theorem (Whitney 1934)

C^m extension of $f \iff$ Taylor's theorem holds on K

The classical Whitney Extension Theorem

Theorem (Whitney 1934)

Suppose $K \subset \mathbb{R}^n$ is compact and $\{f_\alpha\}_{|\alpha| \leq m}$ are continuous real valued functions on K.

There is some $F \in C^m(\mathbb{R}^n)$ with $F|_K = f_0$ and $\partial^\alpha F|_K = f_\alpha$ if and only if

$$\left| f_\alpha(x) - \sum_{|\beta| \leq m - |\alpha|} \frac{f_{\alpha+\beta}(y)}{\beta!}(x-y)^\beta \right| = o(|x-y|^{m-|\alpha|})$$

uniformly for any $x, y \in K$ and any $|\alpha| \leq m$.

Thinking about extensions

Thought experiment

Imagine that you are a drone pilot in training.

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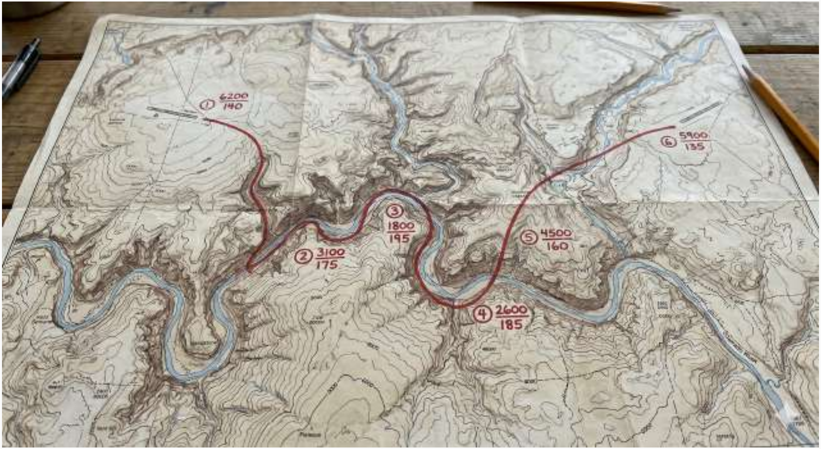

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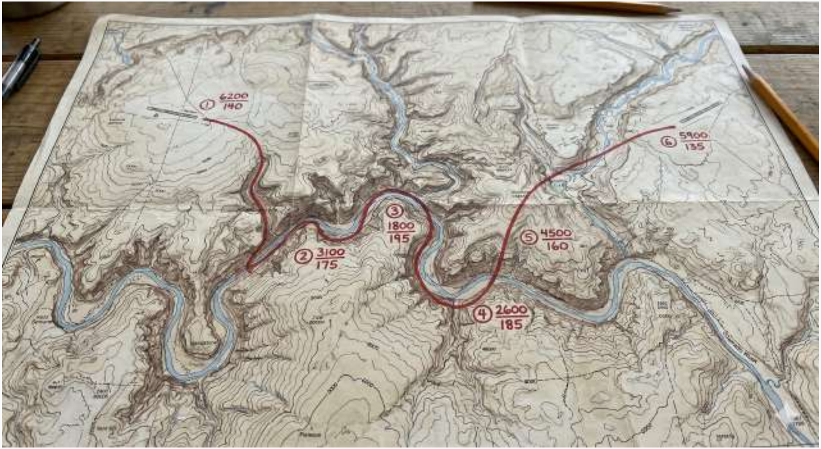

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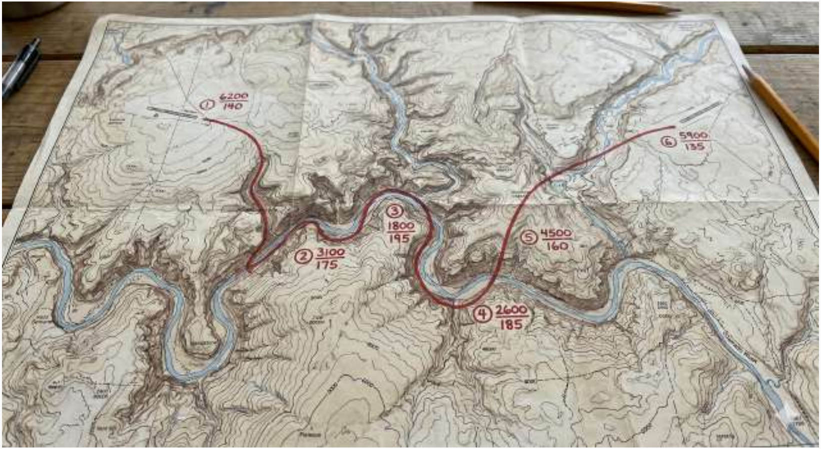

This time, your motion is controlled by the data!

Another thought experiment

How do you drive to the roof of a parking garage?

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We describe motion in $\mathbb{R}^n$ using vector fields: $X : \mathbb{R}^n \to \mathbb{R}^n$.

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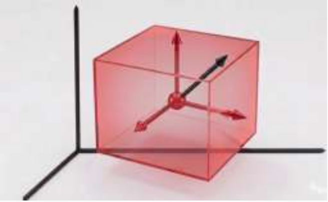

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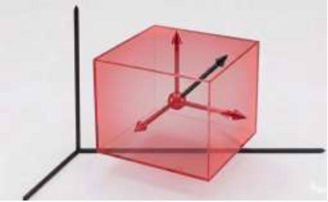
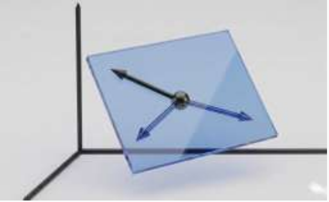


To control motion in $\mathbb{R}^n$, we can
restrict our attention to a **subspace** of vector fields.



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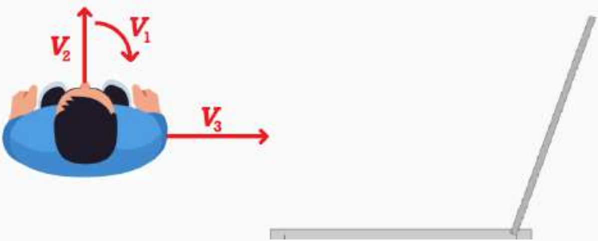

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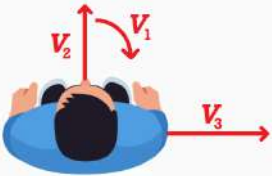

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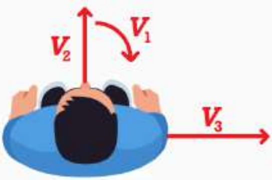

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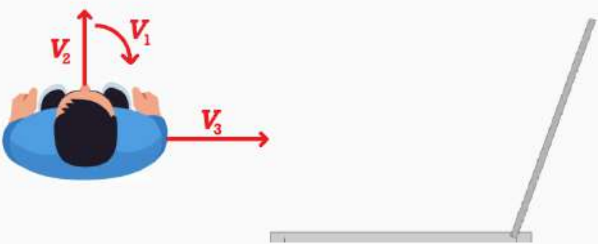

$$V_1 = \frac{\partial}{\partial\theta} \qquad V_2 = \cos(\theta)\,\frac{\partial}{\partial x} + \sin(\theta)\,\frac{\partial}{\partial y} \qquad V_3 = \sin(\theta)\frac{\partial}{\partial x} - \cos(\theta)\frac{\partial}{\partial y}$$

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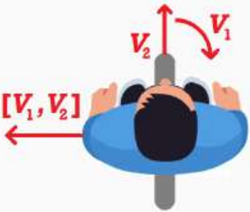

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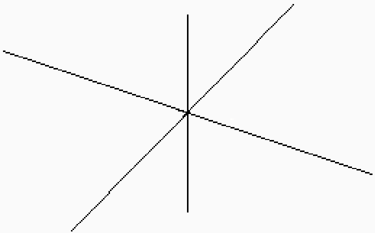
Carnot groups

Write $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_s}$.



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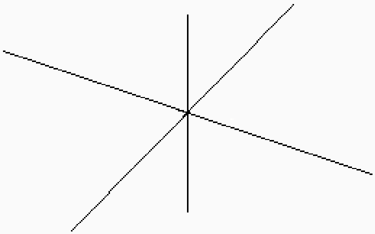

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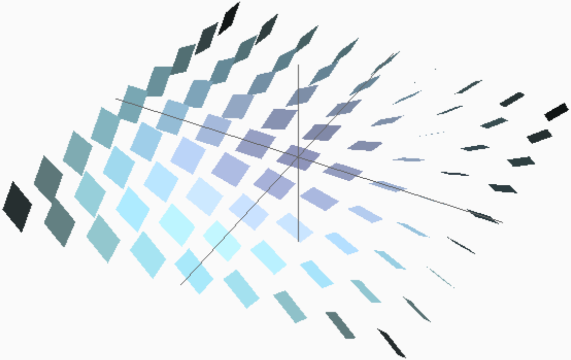

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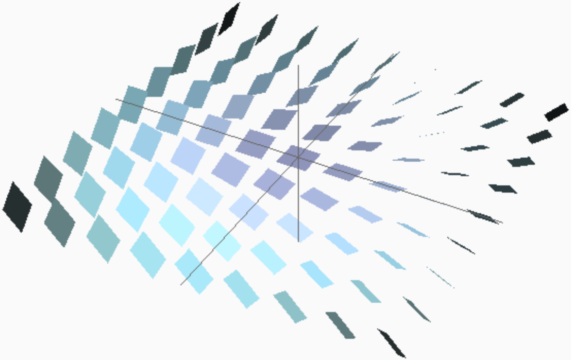

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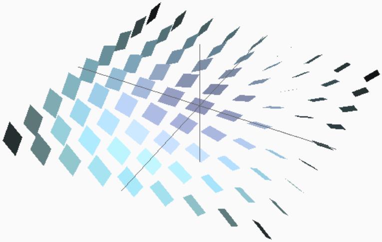
The only paths we will consider are those that lie tangent to this subspace!

Carnot groups



Definition: Given a Lie group product $*$ on $\mathbb{R}^n$, we call $(\mathbb{R}^n, *)$ a Carnot group if

1. $(x_1, x_2, \ldots, x_s) \mapsto (tx_1, t^2 x_2, \ldots, t^s x_s)$ is a group automorphism
2. and any left invariant vector field on $\mathbb{R}^n$ can be written as a linear combination of Lie brackets of the left invariant vector fields $\{X_1, \ldots, X_{n_1}\}$ generated by $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{n_1}}$.

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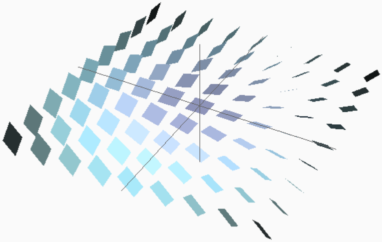

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Translation: We can move in **any** direction by combining movements in **only** the n_1 directions $\{X_1, \ldots, X_{n_1}\}$!

Metric space

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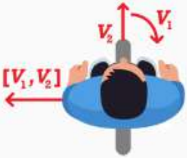

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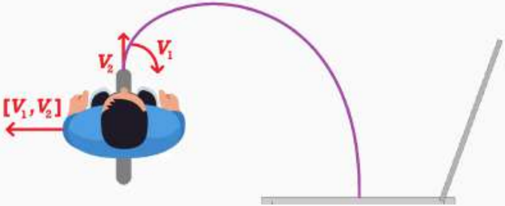

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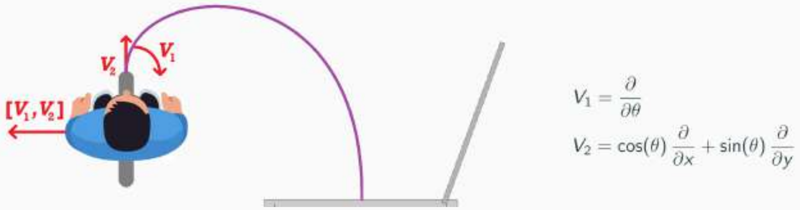

We can define a metric on $\mathbb{R}^n$ by minimizing over lengths of these curves.

The sub-Riemannian Heisenberg group

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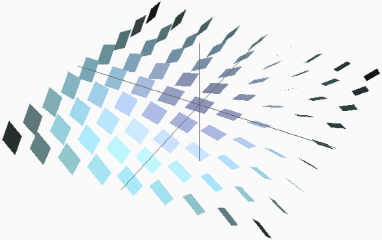

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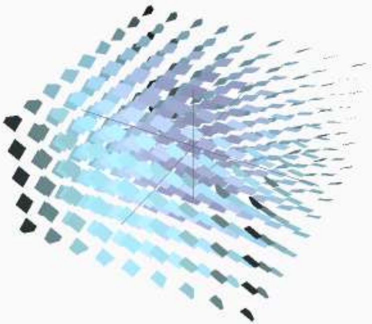

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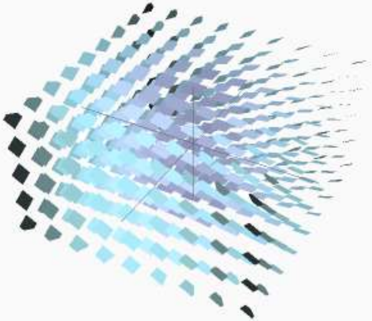

Choose the Riemannian metric that makes X, Y, and $\frac{\partial}{\partial z}$ orthonormal.

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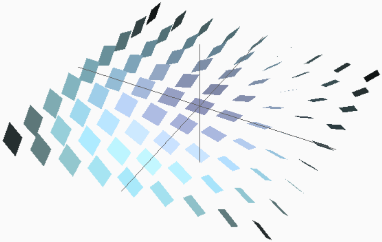

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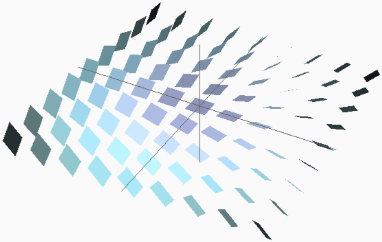

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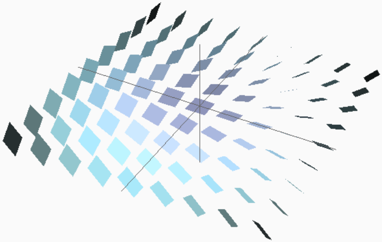

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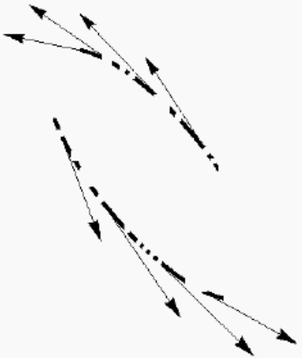
Also, $[X, Y] = -4Z$. All of this implies that $\mathbf{H}$ is a Carnot group.

Whitney extensions in the Heisenberg group

The C^1 case



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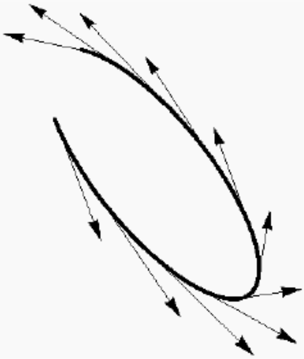

Assume we have continuous maps
$\gamma : K \to \mathbf{H}$ and $\gamma_1 : K \to \mathbf{H}$.

Theorem (Whitney 1934)

C^1 extension of γ

$$\Updownarrow$$

uniform convergence on K of

$$\frac{\gamma(x+h)-\gamma(x)}{h} \to \gamma_1(x)$$

Definition

The **Pansu derivative** at $x \in \mathbb{R}$ of a horizontal curve γ in **H** is

$$\lim_{h \to 0} \delta_{1/h} \left(\gamma(x)^{-1} * \gamma(x+h) \right)$$

whenever this limit exists.

The Pansu derivative is the limit of a difference quotient!

Whitney's Extension Theorem into H

Theorem (Whitney 1934)

$$C^1 \text{ extension of } \gamma$$

$$\Updownarrow$$

uniform convergence of difference quotient to γ' *on* K

Whitney's Extension Theorem into H

Theorem (Whitney 1934)

C^1 extension of γ

$$\Updownarrow$$

uniform convergence of difference quotient to γ' on K

Theorem (SZ 2017)

C^1 horizontal extension of γ

$$\Updownarrow$$

uniform convergence of Pansu difference quotient to γ' on K

Other results

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- γ has a C^m horizontal extension if and only if a particular Taylor-like polynomial of maps on K is $o(|x-y|^{2m})$. [Pinamonti, Speight, SZ 2019]
- γ has a C^∞ horizontal extension if and only if these Taylor-like polynomials are $o(|x-y|^{2m})$ at the same rate. [Pinamonti, Speight, SZ 2023]

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A set E in a metric space X is ______ *rectifiable (by curves)* if there is a family $\{\gamma_i\}$ of ______ curves $\gamma_i : [0,1] \to X$ such that

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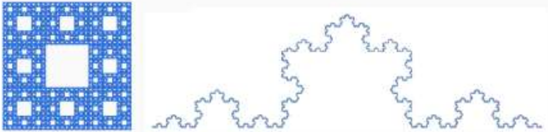
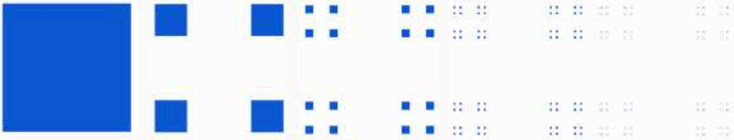
1-rectifiable sets (image credit Nora and Lydia Zimmerman)



Purely 1-unrectifiable sets (because they contain no finite length curves)



Garnet's 4-corner dust: 1-dimensional, but purely 1-unrectifiable

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What about in Carnot groups?

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(For sets with dimension > 1, this has been well studied.
The notions are very different!)

Lusin approximations

The equivalence of Lipschitz and smooth rectifiability in $\mathbb{R}^n$ is due to the following (which itself is a consequence of Whitney's Extension Theorem):

Theorem

Suppose $\gamma : [0,1] \to \mathbb{R}^n$ *is a Lipschitz curve.*
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Translation: Every Lipschitz curve in $\mathbb{R}^n$ is "mostly" smooth.

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Using our Whitney-type result in **H**, we have the same:

Theorem (SZ (2016), Speight (2016), Juliet, Sigalotti (2017))

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Suppose $\gamma : [0,1] \to \mathbf{E}$ *is a Lipschitz curve. For every* $\varepsilon > 0$*, there is a Lipschitz curve* $\gamma : [0,1] \to \mathbb{G}$ *such that*

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for all C^1_H *curves* $\Gamma : [0,1] \to \mathbf{E}$.

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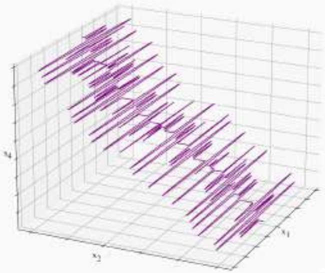
E is the Engel group (which is 1 "step" more complicated than the Heisenberg group).

Corollary: **E** does not have Whitney extensions in the same way as **H**!

Lusin approximations

- Every Lipschitz curve in $\mathbb{R}^n$ is “mostly” smooth.
- Every Lipschitz curve in **H** is “mostly” smooth.
- There is a Lip. curve in **E** that is “mostly” non-smooth.



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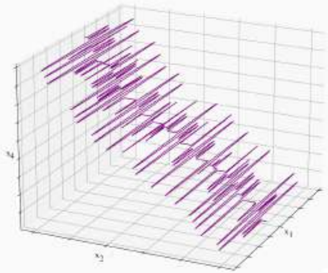

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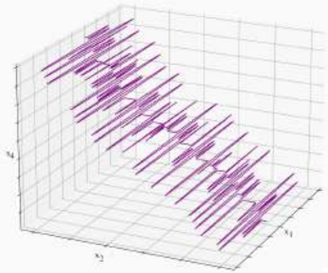

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Answer: No!

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In $\mathbb{R}^n$, Lipschitz rectifiability and C^1 rectifiability are equivalent, but, in such a $\mathbb{G}$, these ideas would be wildly different!

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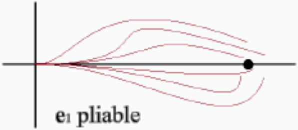

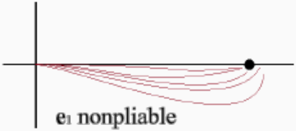

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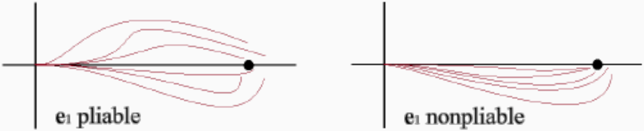

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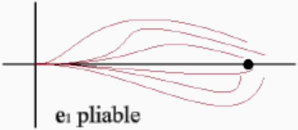

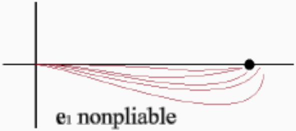

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Future work:
Higher dimensional domains

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